THE LOGICAL FORM OF GEOMETRICAL THEOREMS.

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In the introduction to the Syllabus of Geometry prepared by the Geometrical Association of England, there is a portion devoted to the logic of the subject; so also in the introduction to Halsted's Elements of Geometry. The Association represents the typical theorem by

If A is B, then C is D;

while Halsted takes

x is y.

I have looked at this subject from the point of view of the *Algebra of Logic*.* The typical form which it suggests, has this great advantage over the preceding, that we can, by means of it, prove the truth of the Rules of Contraposition and Conversion. Let A denote the subject of the theorem (as triangle), x the attribute of the antecedent, y the attribute of the consequent; then the typical theorem is

$$\Sigma A \{x = xy\}, \tag{1}$$

where Σ is used to denote *all*.

The contrapositive is

$$\Sigma A \{1 - y = (1 - y)(1 - x)\}. \tag{2}$$

Now (2) is true when (1) is true, for it is

$$\Sigma A \{1 - y = 1 - y - x + xy\},$$

which, when (1) is true, reduces to an identity.

The converse is
$$\Sigma A \{y = xy\}, \tag{3}$$

which does not follow from (1), unless $\Sigma A \{y = x\}$.

The obverse is

$$\Sigma A \{1 - x = (1 - x)(1 - y)\}. \tag{4}$$

When the antecedent of the theorem involves two independent attributes, the type is

$$\Sigma A \{xy = xyz\}. \tag{5}$$

This form has two contrapositives; namely

$$\Sigma A \{x(1 - z) = x(1 - z)(1 - y)\}, \tag{6}$$

and
$$\Sigma A \{y(1 - z) = y(1 - z)(1 - x)\}, \tag{7}$$

both of which are true, when (5) is true.

* Principles of the *Algebra of Logic*, by A. Macfarlane, D. Sc., Edinburgh, 1879.

This type has also two converses, and two obverses.

In the works referred to, a rule of conversion is enunciated, but not proved; thus (in Syllabus) "If of the hypotheses of a group of demonstrated theorems it can be said that one must be true, and of the conclusions that no two can be true at the same time, then the converse of every theorem of the group will necessarily be true."

Let the group of theorems be

$$\Sigma A \{x_1 = x_1 y_1\}, \tag{8}$$

$$\Sigma A \{x_2 = x_2 y_2\}, \tag{9}$$

$$\Sigma A \{x_3 = x_3 y_3\}; \tag{10}$$

it is also given that $x_1 + x_2 + x_3 = 1;$

and $y_1 y_2 = 0, \quad y_2 y_3 = 0, \quad y_3 y_1 = 0.$

By adding (8), (9), (10), we get

$$\Sigma A \{x_1 + x_2 + x_3 = x_1 y_1 + x_2 y_2 + x_3 y_3\};$$

$$\therefore \quad 1 = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Multiply by y_1, then $y_1 = y_1 x_2.$

Similarly $y_2 = y_2 x_2, \quad \text{and} \quad y_3 = y_3 x_3.$

Hence it follows that

$$\Sigma A \{y_1 = y_1 x_1\},$$

$$\Sigma A \{y_2 = y_2 x_2\},$$

$$\Sigma A \{y_3 = y_3 x_3\}.$$